# Differential Geometry

Midterm Examination

September 11, 2025

**Instructions:** All questions carry ten marks.

1. State and prove the Mean Value Theorem for a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$.

2. Let $S_0 = \{(x, y, z) \mid x^2 + y^2 = 1, |z| < 1\}$ denote the open cylinder in $\mathbb{R}^3$. Prove that there exists a diffeomorphism that is *equiareal* between $S_0$ and the unit sphere $S^2$ without a pair of antipodal points.

3. Define the Gauss map for an oriented surface $S$. Prove that at each of point $p \in S$, the differential of the Gauss map is a self-adjoint operator on the tangent space $T_p(S)$ of the surface at $p$.

4. Define an *umbilical* point of an oriented surface $S \subset \mathbb{R}^3$. If a connected oriented surface in $\mathbb{R}^3$ has only umbilical points, then prove that it is contained in a sphere or a plane.

5. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and $S$ denote the regular surface in $\mathbb{R}^3$ given by its graph. Prove that the Gaussian curvature of $S$ is given by the formula

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}$$

where $x, y$ are the corrdinates of $\mathbb{R}^2$.